Excitation of Complex Profiles by CARVE Sequence: Accounting for Spectral Dispersion and Relaxation

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ABSTRACT: Completely Arbitrary Regional Volume Excitation (CARVE) is a technique that enables the excitation of complex shapes in two and three dimensions. It is based on the train of interleaved short RF pulses and gradient steps. For on-resonance magnetization, the CARVE parameters (amplitudes and phases of short RF pulses, and gradient magnitudes and directions) required to flip the magnetization from its initial orientation by 90° within the selected excitation profile are easily calculated. The main advantage of CARVE relative to other techniques for spatial selective excitation is its flexible sequence design which results in compact sequences with low gradient load. The basic CARVE theory is limited to a single, on-resonance, nonrelaxing magnetization. In this paper, we provide the full theory of CARVE including the off-resonance and relaxation effects and verify it experimentally in two and three dimensions. Based on the linearity of the sequence, we propose methods to eliminate undesirable effects and tailor CARVE for any particular application. © 1999 John Wiley & Sons, Inc. Int J Imaging Syst Technol, 10, 225–241, 1999

I. INTRODUCTION

Owing to its numerous well-characterized parameters, nuclear magnetic resonance (NMR) is a versatile tool for studying a variety of systems ranging from free atomic nuclei to the human body (Abragam, 1961; Ernst et al., 1990; Callaghan, 1991). An important property of NMR, especially when applied to living tissues, is its noninvasive character. Unfortunately, closely related to this virtue is poor sensitivity, its principal disadvantage. An obvious way to improve sensitivity is to acquire the signal from the largest representative region of interest (ROI). For example, in NMR spectroscopy of the human body, the ROI may be an organ, body part, or any other irregularly shaped region. In such cases, shapes having cubic, spherical, or cylindrical geometries that can be excited using well-established techniques are usually not representative of the complex shape of the organ or body part under investigation. This leads to a loss in sensitivity (when the actual ROI is smaller than the desired region) or to signal overlap from the surroundings (if the ROI is larger than the desired region).

Numerous methods for volume selective excitation have been proposed but most of them have been limited to simple shapes, for example, slices, cubes, and spheres (Aue et al., 1984; Frahm and Haenick, 1984; Bottomley, 1984; Mueller et al., 1985; Doddrell et al., 1986a,b). The versatility of any method depends on the sophistication of its building blocks. For example, if the building block excites a slice, then three noncollinear slices excite parallelepipeds, three orthogonal slices excite a cuboid, and rotating slices excite either a cylinder or a sphere. Obviously, excitation of nontrivial shapes requires a method with a more sophisticated building block.

We proposed one such method, CARVE, recently (Sersen and Macura, 1996, 1997). Building blocks in CARVE are harmonic functions which provide a much higher freedom in defining the excitation profile compared to other techniques. It is well known from Fourier analysis that any well-behaved function (excitation profile) can be decomposed into an infinite sum of harmonics. Other methods have utilized harmonic functions as building blocks too (Meyer et al., 1990; Spielman et al., 1991; Morrell and Macovski, 1997). The principal difference between CARVE and the other harmonic-based techniques is that they operate in continuous space and, thus, use all the harmonics. This makes them long and demanding on the gradient hardware. CARVE, however, operates in discrete space which limits the number of harmonics used and leads to short, compact excitation sequences. The use of a limited number of harmonics results in an excitation profile that is an approximation of the ideal target profile. However, by proper selection of the number of harmonics and their weights, an approximation with a desired degree of similarity can always be generated (Sersen and Macura, 1997). Another important difference between CARVE and other methods is in gradient sequence design. Whereas other techniques use the same gradient sequence (the k-space trajectory is always the same, spiral, sequential, etc.), CARVE uses a gradient sequence that is custom designed for each profile shape and, in general, corresponds to random walk in k-space (Scheffer and Hennig, 1996). The customization can be directed at minimizing distortions from one or more sources; for example, the gradient sequence can be optimized to minimize the gradient load. This reduces the profile distortions caused by the limited slew rate or the nonlinearity of the gradient unit (Sersen and Macura, 1997).

In this article, we recast the theory of the CARVE sequence and describe a procedure for practical CARVE sequence design and gradient optimization. We analyze the influence of off-resonance and relaxation effects on excitation profile distortion and describe methods for their elimination. The feasibility of CARVE is then
demonstrated experimentally in two and three dimensions on water and water/acetone phantoms.

II. BASIC CARVE THEORY

Three-dimensional NMR imaging methods reveal the spatial distribution of selected NMR or molecular parameters (spin density, longitudinal or transverse relaxation time, and diffusion coefficient) within a nonhomogeneous sample. If, in addition to spatial distribution, a selected property is distributed in resonance frequencies, then a complete description would require a four-dimensional experiment. For example, spin density \( \rho \) can be distributed in space, \( \rho (r) \), but if more than one chemical species exists, it is also distributed in resonance frequencies, \( \rho (r, \omega) \). Because four-dimensional experiments are rather time consuming, it is more efficient to record images at a preselected resonance frequency, \( \rho (r, \omega = \omega_0) \), or to record spectra from the ROI where the spatial distribution of spin density is irrelevant, \( \rho (\omega) \).

The principal goal of the CARVE sequence is to excite magnetization selectively from an ROI which is defined from another prerecorded image. Because all computer-generated images are discrete (i.e., desired excitation profiles are discrete), it is natural to design an excitation sequence which operates in a discrete manner. Thus, instead of a continuous RF and gradient shape function, a discrete rectangular RF and gradient pulses should be used. Consequently, the building blocks of the CARVE sequence is a short RF pulse (with flip angle \( \theta \) and phase \( \varphi \)) and constant gradient pulse \( G \) of duration \( \Delta t \). The effect of the RF and gradient pulses on the equilibrium magnetization

\[
M_0 = \begin{bmatrix} 0 \\ 0 \\ M_0 \rho(r) \end{bmatrix}
\]

can be conveniently described over rotational matrices (Morris and Freeman, 1978). Initially we consider a single frequency, on-resonance, nonrelaxing magnetization. The CARVE sequence with \( N \) pulse pairs (events), \((\theta_1, \varphi_1), G_1(-\theta_2, \varphi_2), G_2 - \ldots - (\theta_n, \varphi_n), G_n\), rotates the equilibrium magnetization into a new position \( M(r) \)

\[
M(r) = R_1(\alpha_0)R_\omega(\theta_0, \varphi_0) \cdots R_1(\alpha_n)R_\omega(\theta_n, \varphi_n)M_0(r).
\]

The gradient pulse rotates the magnetization around the \( z \)-axis exclusively, whereas the RF pulse, depending on the phase, tips the magnetization toward or away from the \( z \)-axis. An angle \( \alpha_i \) for which the magnetization component at \( r \) rotates around the \( z \)-axis (in a time \( \Delta t \) under the influence of the gradient \( G_j \)) is

\[
\alpha_i = \gamma \Delta t r G_i
\]

and \( R_j \) is the transformation matrix which describes this rotation. The rotation by an RF pulse with tilt angle \( \theta \) and phase \( \varphi \) is described by the transformation matrix \( R_{\omega}(\theta, \varphi) \). For consistency with the basic NMR equation \( (\omega_1 = -\gamma \delta_1) \), the direction of the rotation is reversed and the transformation matrices are

\[
R_\omega(\alpha) = \begin{bmatrix} \cos(\alpha) & \sin(\alpha) & 0 \\ -\sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix},
\]

\[
R_\omega(\theta, \varphi) = \begin{bmatrix} \cos(\varphi) & -\sin(\varphi) & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

\[
= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{bmatrix}
\]

\[
= \begin{bmatrix} \cos(\varphi) & \sin(\varphi) & 0 \\ -\sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\]

A. Small-Tip-Angle Approximation. Equation (1) is an exact expression for rotation of isolated spins during the CARVE sequence. Despite the introduced simplifications, Equation (1) is still too complex for further theoretical consideration. A small-tip-angle approximation (Morris and Freeman, 1978; Pauly et al., 1989) offers further simplification but requires short RF pulses (tip angles less than 10°). Then, \( \cos(\theta) \approx 1, \sin(\theta) \approx \theta \) and

\[
R_\omega(\theta, \varphi) \approx \begin{bmatrix} 1 & 0 & -\theta \sin(\varphi) \\ \theta \sin(\varphi) & 1 & -\theta \cos(\varphi) \\ \theta \cos(\varphi) & -\theta \sin(\varphi) & 1 \end{bmatrix}.
\]

Then, a single event (RF pulse + gradient pulse) CARVE rotation matrix \( R_C(\theta, \varphi, \alpha) \) is

\[
R_C(\alpha, \theta, \varphi) = \begin{bmatrix} \cos(\theta) & \sin(\theta) & \theta \sin(\varphi - \alpha) \\ -\sin(\theta) & \cos(\theta) & \theta \cos(\varphi - \alpha) \\ \theta \cos(\varphi - \alpha) & \theta \sin(\varphi - \alpha) & 1 \end{bmatrix}.
\]

According to Equation (1), the equilibrium magnetization is rotated into a new position after the single event,

\[
M_1(r) = R_C(\theta_1, \varphi_1, \alpha_1)M_0(r) = \begin{bmatrix} \theta \sin(\varphi - \alpha) \\ \theta \cos(\varphi - \alpha) \\ 1 \end{bmatrix}M_0(r)
\]

An RF pulse in a single CARVE event tips the equilibrium magnetization by \( \theta \) from the \( z \)-axis along the direction rotated by \( \varphi \) from the \( z \)-axis and the gradient rotates it by \( \alpha \) around the \( z \)-axis. It is important to note that \( \theta \) and \( \varphi \) operate uniformly on all spatial components of the magnetization, whereas rotation by \( \alpha \) is spatially dependent, Equation (2). After two CARVE events

\[
M_2(r) = R_C(\theta_2, \varphi_2, \alpha_2)R_C(\theta_1, \varphi_1, \alpha_1)M_0(r)
\]

\[
= R_C(\theta_2, \varphi_2, \alpha_2)M_1(r)
\]

and explicitly by neglecting higher order terms in \( \theta \),

\[
M_{2a}(r) = -M_0(r)[\theta_1 \sin(\varphi_1 - \alpha_1 - \alpha_2) + \theta_2 \sin(\varphi_2 - \alpha_2)],
\]

\[
M_{2b}(r) = M_0(r)[\theta_1 \cos(\varphi_1 - \alpha_1 - \alpha_2) + \theta_2 \cos(\varphi_2 - \alpha_2)],
\]

\[
M_{2c}(r) = M_0(r)
\]

The effects of two successive events co-add with the phase factors \( \varphi_1, \varphi_2, \theta_1, \) and \( \theta_2 \). Depending on their values, the resulting tip angle
can be anywhere between \(\theta_i - \theta_j\) and \(\theta_i + \theta_j\). Because the angles \(\alpha\) are spatially dependent, the resulting tip angle also depends on the spatial coordinates. By a judicious choice of phases and tip angles, one can achieve tip angle co-addition within the ROI and subtraction outside. However, transition from addition to subtraction obviously takes place smoothly and with just two events a border of the ROI is ill defined. For better definition of the excitation profile, more events are needed. By repeating the same procedure shown in Equations (6) and (7), it is easy to obtain expressions for the spatial distribution of the magnetization components after \(N\) events:

\[
M_{N,\alpha}(r) = -M_{\theta} \rho(r) \sum_{i=1}^{N} \theta_i \sin \left( \varphi_i - \sum_{j=1}^{N} \alpha_j \right),
\]

\[
M_{N,\beta}(r) = M_{\theta} \rho(r) \sum_{i=1}^{N} \theta_i \cos \left( \varphi_i - \sum_{j=1}^{N} \alpha_j \right),
\]

\[
M_{N}(r) = M_{\theta} \rho(r).
\] (9)

For the small-tip-angle approximation, the longitudinal magnetization is invariant. Because it is unobservable, it can be safely neglected in further analysis. The small-tip-angle approximation also distorts the transverse component. However, these distortions are proportional and thus the ratio of \(x\) and \(y\)-components is exact. Then, using a complex number notation, the transverse component can be represented as

\[
M_x(r) = M_{N,x} + iM_{N,y} = iM_{\theta} \rho(r) \sum_{i=1}^{N} \Theta_i \exp \left( -i\gamma \Delta t \sum_{j=1}^{N} G_j \right).
\] (10)

where we have also introduced a complex RF pulse

\[
\Theta_i = \theta_i \exp(i\varphi_i).
\] (11)

Equation (10) is a discrete form of the well-known small-tip-angle approximation (Pauly et al., 1989) and is exact up to the constant scaling factor.

B. Ideal Profile and Complex RF Pulses. Defining a k-space vector (Callaghan, 1991) as

\[
k_i = -\gamma \Delta t \sum_{j=1}^{N} G_j \]

Equation (10) simplifies to

\[
M_x(r) = iM_{\theta} \rho(r) \sum_{i=1}^{N} \Theta_i \exp(+irk_i).
\] (13)

Equation (13) describes the spatial dependence of the transverse magnetization after the CARVE sequence with \(N\) events (RF pulses \(\Theta_i\) and gradient pulses \(G_j\)). The excitation profile of the CARVE sequences \(P_x(r)\) is obtained by normalizing Equation (13) with the spatial distribution of equilibrium magnetization in a homogenous sample, \(\rho(r) = 1\):

\[
P_x(r) = \frac{M_x^0}{iM_{\theta}} = \sum_{i=1}^{N} \Theta_i \exp(+irk_i).
\] (14)

Thus, the excitation profile, \(P_x(r)\), is a weighted sum of \(N\) harmonic functions. From Fourier analysis a function, say, \(P_x(r)\), can be decomposed into an infinite set of harmonic functions

\[
P_x(r) = \sum_{i=1}^{\infty} \Theta_i \exp(+irk_i).
\] (15)

where \(\Theta_i\) are Fourier coefficients. From Equations (14) and (15) it follows that the actual excitation profile, \(P_x(r)\), is built from the CARVE sequence elements, \(\Theta_i\), \(k_i\), in the same manner as an ideal excitation profile \(P_x(r)\) is built from the Fourier coefficients \(\Theta_i\) and k-space vectors. It is useful to extend the analogy of the CARVE sequence and Fourier analysis. By noting that the real space and k-space form Fourier pairs of spaces, then the real space coordinates \(r\) and the k-space coordinates \(k\) are Fourier pairs. Thus, any ideal profile can have its representation in both real space and in k-space. If the ideal profile \(P_x(r)\) in real space is confined in the \(D\)-dimensional box with side \(L\) (definition space \(\mathbb{X} = [-L/2, L/2]^D\)), we then obtain its representation in k-space by inverting Equation (15):

\[
\Theta(k_i) = \frac{1}{L^D} \int_0^L \exp(-irk_i) P_x(r) dr.
\] (16)

Thus, the representation of an ideal profile in k-space is equal to the elements of an infinitely long CARVE sequence. To define the excitation profile it is convenient to use harmonic functions that form an orthogonal set of functions in real space \(\mathbb{X}\). This imposes the following restriction to the vectors in k-space

\[
k_i = \frac{2\pi}{L} m_i; \quad m_i \in \mathbb{Z}^D.
\] (17)

where \(m_i\) denotes a \(D\)-dimensional integer vector. In practice, an ideal profile is already in a discrete form. If this is represented by a \(D\)-dimensional matrix of size \(M^0\) at equidistant points

\[
P_x = P_x \left( \frac{Lu}{M} \right); \quad u = (u_1, \ldots, u_D);
\]

\[
u_1, \ldots, u_D = \frac{M}{2}, \ldots, \frac{M}{2} \quad (18)
\]

then Equation (16) becomes

\[
\Theta_i = \frac{1}{M^0} \sum_{m_i = -M/2}^{M/2-1} \sum_{m_j = -M/2}^{M/2-1} \exp \left( -2\pi i m_i u / M \right) P_x
\] (19)

and its inverse:
\[ P_r = \sum_{m=-M/2}^{M/2-1} \sum_{n=-M/2}^{M/2-1} \exp \left( \frac{2\pi i m u}{M} \right) \Theta_n \]  
(20)

where

\[ \Theta_n = \begin{cases} 
\Theta_m & l \in 1, \ldots, N \\
0 & l \not\in 1, \ldots, N 
\end{cases} \]

Equations (19) and (20) establish a fundamental relationship between the shape of exited profile \( P \) and the CARVE sequence parameters \( \Theta \).

C. The Profile Approximation. The CARVE sequence is composed of a finite number of elements and is thus an approximation of the ideal excitation profile. An adequate approximation of an ideal excitation profile can be realized with a finite number of pulses by selecting a suitable projection method.

It is convenient to express an ideal excitation profile \( P_c \) as a sum of the actual profile \( P_r \) and residual \( R_r \):

\[ P_c(r) = P_r(r) + R_r(r) \]  
(21)

As a measure of the goodness of the approximation, the profile norm is used. Then, the best approximation is that which minimizes the squared norm of the residual \( \| R_r \|^2 \). By Parseval's relation, the squared norm of the profile is the same in both real and k-space:

\[ \sum_{i=1}^{L} |(\Theta)|^2 = \| P_r \|^2. \]  
(22)

Since the norm of an ideal profile is constant,

\[ \| P_c \|^2 = \| P_r \|^2 + \| R_r \|^2 \]  
(23)

the norm of the residual is minimized when the norm of the actual profile is maximized. To quantify the goodness of the approximation, we introduce a resemblance factor \( \eta \) \((\eta \in (0,1))\) which is the ratio of the squared CARVE norm and the ideal excitation profile norm.

\[ \eta = \frac{\| P_r \|^2}{\sum_{i=1}^{N} |(\Theta)|^2} \]  
(24)

Obviously, the approximation becomes better as \( \eta \to 1 \). For the \( N \)-pulse CARVE sequence, the best approximation is obtained by selecting \( N \) \( \Theta \) coefficients [out of \( M^2 \) obtained from Equation (19)] with the highest absolute values of \( \Theta \)s. The k-space points that contain the selected \( \Theta \) coefficients single out the harmonic functions on which an ideal profile has the largest projection. These k-space points are subsequently used to calculate gradient sequence.

D. CARVE Sequence Design. Ideally, the CARVE sequence consists of interleaved rectangular gradient pulses (with amplitude \( G_j \) and duration \( \Delta r \)) and infinitely short RF pulses. Fast gradient pulsing (required to minimize relaxation and off-resonance effects) is hard to implement, and a continuous stream of gradients (gradient steps) is used instead. If the small-tip-angle pulse duration, \( t_p \), is negligible compared to \( \Delta t \), \((t_p \ll \Delta t)\) and the frequency offset induced by the maximum gradient pulse is smaller than the pulse bandwidth \((2\pi t_p \gg \gamma G_{\text{max}} L)\), the gradient effects during the RF pulse can be neglected. In practice, both conditions are easy to fulfill and the gradient pulse sequence is a step function defined by the gradient step unit \( G_0 \) and an integer vector \( n \):

\[ G_j = G_0 n_j. \]  
(25)

The gradient step unit is a constant of the experimental conditions and can be calculated by substituting Equation (25) into Equations (12) and (17):

\[ G_0 = \frac{2\pi}{\gamma \Delta t L}. \]  
(26)

From the same substitution, we find a condition that allows the calculation of the integer vectors \( n \):

\[ m_j = -\sum_{p=1}^{N} n_p. \]  
(27)

From Equation (17), a sequence of selected k-space points \( k_1, -k_2, \ldots, -k_N \) determines a sequence of k-space integer vectors \( m_1, -m_2, \ldots, -m_N \). The sequence of gradient integer vectors \( n_1, -n_2, \ldots, -n_N \) can be calculated from the differences of Equation (27)

\[ n_N = -m_N \]

\[ n_l = m_{l+1} - m_l, \quad l = N - 1, \ldots, 1. \]  
(28)

Equations (25) and (28) imply that the gradient sequence depends on the order in which the selected k-space points are visited. Thus, a gradient sequence with the desired properties can be designed by a suitable selection of the k-space walk. Theoretically, every walk through selected k-space points produces the CARVE profile. In practice, some walks are better than others because of the reduction in the load placed on the gradient hardware. Because each k-space point is visited only once, for \( N \) points there are \( N! \) k-space walks. Among them, the best is a walk with minimum gradient load. The total number of walks can be prohibitively large, (for \( N = 100, N! \approx 10^{156} \)), and a walk with a minimal gradient load cannot be found by a systematic search. Such a walk can, however, be found using optimization techniques, for example, simulated annealing (Metropolis et al., 1953; Kirkpatrick et al., 1988). The walk is found by minimizing a penalty function which expresses the gradient hardware load.

To excite a D-dimensional profile \((D = 2, 3)\) for each CARVE event (out of \( N_i \)) an RF pulse angle, \( \theta \), phase \( \varphi \), and \( D \) orthogonal gradients \( G^D \) need to be calculated. This is illustrated in Figure 1 for an A-shaped two-dimensional profile. First, for the desired profile [Figure 1(a)], a D-dimensional (here \( D = 2 \)) discrete Fourier transform is performed [Figure 1(b)]. Coefficients \( \Theta \) are then calculated.
Here we focus on the walks that minimize the load on the gradient system. The penalty function $E$ that expresses the gradient load can be written as the normalized sum of the squared gradient magnitudes

$$E = \frac{1}{NG_0^2} \sum_{j=1}^{N} |G_j|^2 = \frac{1}{N} \sum_{j=1}^{N} |n_j|^2 = \frac{1}{N} \sum_{j=1}^{N} \sum_{d=0}^{D} n_{j,d}^* n_{j,d}$$  \hspace{1cm} (29)$$

A simulated annealing protocol is used to find a walk for which the penalty function for a given object and number of events is a minimum. Because this minimum is sought merely to minimize distortions, any walk for which the penalty function is below certain threshold produces a satisfactory excitation profile. Therefore, the minimization can be performed quickly and more than one random walk with similar properties can be easily found. Table 1 shows the values of the penalty function for the typical k-space walks shown in Figure 2. The penalty function is highest for a random unoptimized walk and lowest for an optimized random walk.

**F. How CARVE Works.** During the CARVE sequence, magnetization is subjected to rotations by RF pulses toward or away from the z-axis and rotations by the gradient steps around the z-axis. Because gradient rotation is spatially dependent, every magnetization component has different trajectory during the sequence. However, at the end of the sequence, all components within the profile realign along the y-axis and outside the profile are restored back along z-axis. This is illustrated in Figure 3 where typical trajectories within and outside the excitation profile are shown. The arrows in Figure 3(a) show the position of the magnetization components for which the trajectories are shown in Figure 3(b). Within the profile, the magnetization components are aligned such that the majority of the RF pulses tip them from the z-axis [Figure 3(b), left]. At the same time, the components outside the profile are randomly tipped toward and away from the z-axis so that the net effect is no tilt at all [Figure 3(b), right]. This is even more clearly seen in Figure 3(c) where the absolute value of the transverse magnetization is plotted as a function of time during the sequence. Within the profile, the transverse magnetization mostly grows (left) whereas outside the profile magnetization wanders in the vicinity of the z-axis having a relatively small transverse component (right). Figure 3 also illustrates the vulnerability of the CARVE sequence to external interference. Any effect that rotates the magnetization around the z-axis or changes the magnitude of the transverse magnetization alters the precalculated trajectories and diminishes the difference between the components inside and outside the profile.

**G. Linearity and Distributivity of CARVE.** The principal CARVE expression, Equation (13), shows that the excited profile is a superposition of the respective k-space vectors. Because this superposition is linear for a given set of k-space coefficients, the

<table>
<thead>
<tr>
<th>Walk</th>
<th>Penalty Function, $E$, Eq. (29)</th>
<th>$\sqrt{E}$</th>
<th>$\sqrt{(E/E_{min})}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Random</td>
<td>78.500</td>
<td>8.860</td>
<td>5.470</td>
</tr>
<tr>
<td>Sequential</td>
<td>5.400</td>
<td>2.323</td>
<td>1.435</td>
</tr>
<tr>
<td>Random optimized</td>
<td>2.625</td>
<td>1.620</td>
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</tbody>
</table>

Table 1. Penalty function for k-space walks from Figure 2.

E. Finding an Optimal Walk. The criterion for selecting the coefficients $i_d$ does not specify the sequence in which the k-space points are visited. Neglecting empirical distortions, all k-space walks are equivalent and produce the same excitation profile. However, each k-space walk is sensitive to the experimental conditions and, in practice, produce different distortions of the actual profile. Thus, the actual k-space walk that is chosen depends on the experimental parameters for which the walk is optimized.
same profile is obtained irrespective of the order of summation or regrouping of the coefficients. This property has already been used to select a walk with the desired properties. Once the walk was selected, the profile was reconstructed in a single sequence of $N$-events. Equation (13) does not specify whether the summation terms should come from one or more experiments but specifies their number, exact positions in k-space, and values. This means that the selected coefficients can be arbitrarily distributed among $N_x$ independent experiments whose sum always produces the same profile. The distributivity of CARVE provides an additional degree of freedom in designing a sequence having the desired properties. For example, distortions of the excitation profile that are proportional to the sequence length can be eliminated simply by distributing a long sequence into a number of shorter sequences. Figure 4 illustrates this on a 100-event sequence that excites the A-shaped profile. The original sequence of 100 events ($N = 100$; Figure 4(a)) can be distributed into ten 10-event sequences ($N_x = 10, N_y = 10$; Figure 4(b)) or into 100 one-event sequences ($N_x = 100, N_y = 1$; Figure 4(c)). In the absence of imperfections, the sum of 10 subprofiles obtained by the sequences in Figure 4(b) or the sum of 100 subprofiles from sequences in Figure 4(c) gives the same profile as the 100-event sequence from Figure 4(a). Distributivity of the sequence is particularly important for excitation of three-dimensional profiles where even for a modest spatial resolution a large number of coefficients is needed. For example, 100 coefficients approximate a two-dimensional object with resolution of 10 coefficients per dimension. However, for a three-dimensional object with the same resolution, 1,000 coefficients ($10^4$) are needed.

**H. Three-Dimensional CARVE.** The extension of CARVE from two to three dimensions requires a larger number of coefficients and an additional gradient component, $G_z$ (Figure 5). A detailed description of three-dimensional CARVE is presented elsewhere (Serša and Macura, 1998).

**III. INFLUENCE OF OFF-RESONANCE AND RELAXATION**

The basic CARVE theory presented so far deals with isolated, nonrelaxing magnetization with a single well-defined resonance frequency $\omega_0$. Such a system is easy to realize practically, but it is of interest primarily for the experimental demonstration of the various aspects of the CARVE implementation. To illustrate the potential usefulness of CARVE, the basic theory needs to account for other detrimental effects, the most notable of which are off-resonance and relaxation effects. In real samples, both effects are present and may seriously distort the experimental excitation profile if not accounted for.

**A. Off-Resonance Effects.** In real samples, instead of a oneresonance frequency for all spins of the same kind, the frequencies are continuously or discretely distributed. The frequency dispersion can be caused by an inhomogeneous magnetic field, spin-spin relaxation, and chemical shift. In the rotating frame fixed to the spectrometer resonance frequency $\omega_0$, off-resonance spins rotate with a difference $\Delta \omega$ between their own frequency and the spectrometer frequency. Thus, during the CARVE sequence, in addition to the rotation caused by the magnetic field gradient, magnetization rotates around the $z$-axis under the influence of the resonance offset $\Delta \omega$. This additional rotation manifests as a phase shift $\beta$. The phase shift is proportional to the resonance offset frequency and at the end of single gradient interval $\Delta t$ is

$$\beta = \Delta \omega \Delta t$$

(30)

In complete analogy to the rotation caused by gradients, off-resonance rotation acts only on the transverse component of the magnetization. Consequently, it can be incorporated into the expression for the transverse magnetization, Equation (10), as a term which adds to the gradient rotation:
\( M^*_z(r) = i M_{\text{ref}}(r) \sum_{i=1}^{N} \Theta \exp\left(-i \Delta t \sum_{j=1}^{N} (\gamma \tau G_j + \Delta \omega)\right) \). (31)

With regard to Equations (12), (14), and (30)

\[ P_{N,\beta}(r) = \sum_{i=1}^{N} \Theta \exp(+i k \cdot r) \exp(-i \beta(N - l + 1)) \] (32)

The last expression gives the excitation profile \( P_{N,\beta}(r) \) in an \( N \) event CARVE sequence in the presence of resonance offset. Because the resonance offset is a function of the sample composition, it is useful to separate off-resonance effects from the parameters of the CARVE sequence. This can be achieved by noting that Equation (32) is a discrete FT of the product \( \Theta \exp(-i \beta(N - l + 1)) \)

\[ P_{N,\beta}(r) = FT[\Theta \exp(-i \beta(N - l + 1))] \] (33)

Then, using the convolution theorem, which states that FT of the product is equal to the convolution of FTs we find

\[ P_{N,\beta}(r) = FT[\Theta] * FT[\exp(-i \beta(N - l + 1))] \] (33a)

The first Fourier transform, \( FT[\Theta] = P_{N}(r) \), Equation (14) is known. It is convenient to define the second transform as a new function (point spread function) PSF

\[ PSF_{\beta}(r) = \frac{1}{L_0} \sum_{l=1}^{N} \exp(+i k \cdot r) \exp(-i \beta(N - l + 1)) \] (34)

and then

\[ P_{N,\beta}(r) = P_{\beta}(r) * PSF_{\beta}(r) = \int_{x} P_{\beta}(r - r') PSF_{\beta}(r') \, dr' \] (35)
Figure 4. Distributive property of CARVE. Shown are k-space walks and CARVE sequences for the profile from Figure 1(a). Sequences are optimized to have a very low gradient load: (a) single sequence with 100 pulses, (b) Ten sequences of 10 pulses each, and (c) 100 single-pulse sequences.

or in discrete form

\[ P_{n,\beta} = \frac{1}{M^{D}} \sum_{u_1=-M/2}^{M/2-1} \cdots \sum_{u_D=-M/2}^{M/2-1} P_{n-u} \cdot PSF_{n',\beta} \]  \hspace{1cm} (36)

The last two expressions establish the relationship between the profiles with and without resonance offset distortion. The profile in the presence of resonance offset is a convolution of the on-resonance profile and the PSF. Thus, the PSF shows the distortion of a single pixel in the excitation profile caused by resonance offset.

Setting the on-resonance profile equal to a single pixel \( P_{n}(r) = \delta(r) \), we find that the off-resonance profile is equal to the PSF

\[ P_{n,\beta}(r) = PSF_{\beta}(r) \]  \hspace{1cm} (37)

or, in reverse, setting \( PSF_{\beta}(r) = \delta(r) \) we find

\[ P_{n,\beta}(r) = P_{n}(r) \]  \hspace{1cm} (38)

that is, when the PSF approaches Dirac’s delta function off-resonance distortion vanishes. Since the point spread function distorts
the desired profile in an unpredictable manner ($\beta$ is a function of sample composition), the main goal in the actual experimental design is to make the actual PSF as close as possible to the Dirac delta function. In a single frequency system \( PSF(r) = \delta(r) \) when \( \Delta \omega = 0 \), that is, when the profile is generated on-resonance. Owing to the periodicity of the harmonic function, the PSF is equal to Dirac's delta function for all frequencies in which the phase shift $\beta$ is a multiple of $2 \pi$

$$\Delta \omega = \frac{2\pi n}{\Delta t}; \quad n \in \mathbb{Z};$$  (39)

The last condition is useful when the sample has two sharp resonance lines, for example, water and fat. Then, by setting the gradient step duration to $\Delta t = 2\pi/\Delta \omega$ (where $\Delta \omega$ is the line separation) and carrier frequency on one of the lines, \( PSF = \delta(\omega)_{\text{mod} \Delta \omega} \).

A difference between the on-resonance and off-resonance profiles depends also on the k-space walk. Because the walk is random, it is more convenient to analyze off-resonance effects by means of simulations rather than explicit calculations. Figure 6 shows computer simulations of the influence of resonance offset on the excitation profile (character A from Figure 1) and PSF sequential and gradient optimized random walk. In both cases, a series of profiles and PSFs are calculated for $\beta = 0^\circ, 0.625^\circ, 1.250^\circ, \ldots, 9.375^\circ$. As expected, in both cases, the distortions increase with the magnitude of the offset. However, owing to the different structure of the walks, the distortions are very different. In the sequential walk (Figure 6(a)), the profile (and the PSF) are merely shifted along the direction of slowest gradient progression with very little distortion. In the random walk (Figure 6(b)), both the profile and the PSF stay in place but spread around quickly beyond recognition. Both effects are easy to explain qualitatively. In the systematic walk, the phase error due
Figure 6. Off-resonance effects. (a) CARVE profile (top) and PSF (bottom) for the sequential walk [from Figure 2(b)] at evenly increasing off-resonances $\Delta \omega \Delta t = 0^\circ - 9.375^\circ$. Off-resonance causes uniform profile shift proportional to the off-resonance phase, in the direction of the slowest propagation through k-space (here y-axis). The profile (and PSF) also move along the principal scanning direction (here x) but, because of higher propagation, the rate is much smaller. (b) CARVE profile (top) and PSF (bottom) for CARVE sequence with a random walk [from Figure 2(d)] at evenly increasing off-resonances $\Delta \omega \Delta t = 0^\circ - 9.375^\circ$. A random k-space walk introduces severe blurring in excitation profile, even at very low off-resonances. Because there is no preferential direction in the k-space walk, one can imagine that the blurring is due to the shift of the profile elements, simultaneously in all directions.

to the offset builds up steadily and is largest along the axis of slowest passage (here, k_y). Because of the systematic build-up of the error, it manifests as an additional time proportional phase increment which upon FT produces merely a shift of the profile or PSF. In the random walk, there is no preferential direction and the phase increases randomly producing a random modulation through the whole k-space. Consequently, the FT of such a signal produces increased randomness of the profile and the respective PSF. In analogy to the systematic walk, the random walk can be thought of as causing a simultaneous shift of the profile (or PSF) in all directions. While it may appear otherwise, the fact that in the random walk the off-resonance profile spreads quickly into the noise floor is of great advantage. Normally, besides off-resonance there is always an on-resonance profile which is invariant to the structure of the

optimized walk. In the random walk, the off-resonance profile is lost into the noise (leaving the on-resonance profile visible). In the systematic walk, the shifted off-resonance profile may seriously interfere with the on-resonance profile because it remains with comparable magnitude. Apparently, in experiments repeated with slightly different random walks, the off-resonance profile spreads into the noise in different ways and the on-resonance profile remains unaltered. Because the off-resonance effects are uncorrelated, coaddition of $N$ such experiment yields an increase in the on-resonance/off-resonance signal ratio by a factor of $\sqrt{N}$.

**B. Relaxation Effects.** Equilibrium magnetization is longitudinal; thus any transverse magnetization spontaneously decays to zero. Here we assume that relaxation is monoeponential and that the magnetization decays with relaxation time $T_2^*$ through the whole sample. From the CARVE point of view, relaxation is a decrease of transverse magnetization during the sequence, even when no RF pulse is applied. The transverse component created after the $l$-th pulse in the $N$-event CARVE sequence immediately begins to relax and continues through the remaining time of the sequence, $(N-l+1)\Delta t$. Every component relaxes with the same relaxation time during its own time interval. In analogy to Equation (32) where off-resonance effects introduce phase modulation, relaxation effects introduce amplitude modulation in the on-resonance nonrelaxing equation. Then Equation (13) can be written as

$$M_j(r) = iM_0\rho(r) \sum_{i=1}^{N} \Theta_i^{\rho}(\omega) \exp(-(N-l+1)\Delta t/T_2^*)\exp(+i\kappa r).$$

(40)

where $\Theta_i^{\rho}$ is a new set of complex RF pulses that, in the presence of relaxation, produces the same profile as $\Theta_i$ in a nonrelaxing system. Comparing Equations (13) and (40) we find

$$\Theta_i^{\rho} = \Theta_i \exp((N-l+1)\Delta t/T_2^*).$$

(41)

that is, in the presence of relaxation, the original excitation profile can be recovered by exponential weighting of the original (on-resonance, nonrelaxing) complex RF pulses.

**IV. ELIMINATION OF SPECTRAL ARTEFACTS**

In a real, inhomogeneous sample, every pixel may have its own spin density $\rho$ that may have its own spectrum $\rho(r, \omega)$. Because real spectra represent a collection of magnetization components that precess at different resonance frequencies and relax with different relaxation times, both phase and amplitude will be distorted in the CARVE experiment. When distortions also depend on the spatial coordinates, the only way to eliminate them is to shorten the sequence duration. This ultimately leads to a single-pulse CARVE sequence and the profile reconstruction using distriubutority of the sequence (vide supra).

By assuming that each pixel has the same spectrum, the spatial and spectral dependencies can be separated

$$\rho(r, \omega) = \rho(r)\rho(\omega).$$

(42)

Then, the total magnetization can be expressed as an integral over the spectrum

$$M^*_j(r) = iM_0\rho(r) \sum_{i=1}^{N} \Theta_i^{\rho}(\omega) \exp(-i\Delta t \sum_{j=1}^{N} (\gamma G_j + \omega)) \exp(-i(N-l+1)\Delta t \omega) \exp(ik \cdot r).$$

(43)

and by reversing the order of integration and summation

$$M^*_j(r) = iM_0\rho(r) \sum_{i=1}^{N} \Theta_i^{\rho}(\omega) \int_{-\infty}^{\infty} d\omega \rho(\omega) \exp(-i(N-l+1)\Delta t \omega) \exp(ik \cdot r).$$

(44)

By noting that the expression in squared brackets is normalized free induction decay (FID) $S_N/N/\sqrt{S_0}$ at time $(N-l+1)\Delta t$, we obtain

$$M^*_j(r) = iM_0\rho(r) \sum_{i=1}^{N} \Theta_i^{\rho}(\omega) S_{N-l+1} S_0 \exp(ik \cdot r).$$

(44a)

Here $\Theta_i^{\rho}$ represents a complex RF pulse necessary to produce a distortionless profile in the presence of a spatially uniform spectral distribution $\rho(\omega)$. Comparing Equations (44a) and (13), we find that

$$\Theta_i^{\rho} = \Theta_i S_{N-l+1}/S_0.$$  

(45)

Thus, RF pulses in a spectrally compensated CARVE can be obtained by normalizing the original complex pulse (single frequency, on-resonance, nonrelaxing) with the FID intensity at the corresponding point in time. Because the FID is complex, $\Theta_i^{\rho}$ is corrected simultaneously for both phase and amplitude distortions. The two effects can be easily separated, and then the amplitude normalization produces relaxation-compensated RF pulses

$$\Theta_i^{\rho} = \Theta_i S_{N-l+1}/S_0.$$  

(46)

whereas the phase normalization gives off-resonance-compensated RF pulses

$$\Theta_i^{\rho} = \Theta_i S_{N-l+1}/S_0.$$  

(47)

Equations (45) to (47) enable, in principle, the correction of any artefact that is uniform through the sample volume. In practice, these equations are of limited value (vide infra) because an actual FID may have a large number of zero crossings ($S_{N-l+1} = 0$). This makes some of the compensated coefficients grossly distorted and the quality of corrected profile poorer than theoretically predicted. In principle, that can be remedied by judicious choice of the gradient intervals $\Delta_t$ (which do not need to be equal) so that zero crossings of the FID can be avoided.
V. EXPERIMENTAL

To test the various theoretical and practical aspects of the CARVE design, we used a 7 T Bruker AMX 300 high-resolution spectrometer with microimaging accessories. The sample was a 5-mm tube filled with water or a water-acetone mixture. The tube was oriented along the direction of the static magnetic field. The proton spectrum of the water-acetone mixture has two spectral lines which, at a magnetic field strength of 7 T, are approximately 700 Hz apart. The water/acetone ratio was optimized so that both lines had the same intensity. For two-dimensional, x- and y-gradients were used without slice selection because the sample is uniform along the z-axis. CARVE excitation profiles were visualized by standard two or three-dimensional gradient echo imaging sequences in which the initial 90° pulse was replaced by a CARVE sequence. The profiles were imaged with a 5-mm field of view on a 256 × 256 matrix using two scans, 50 kHz spectral width, 1-s repetition time, and an echo time equal to the sequence duration + 8 ms. The RF tip angles were adjusted by varying the pulse duration $t_p = 0.5–5 \mu$s at a constant transmitter gain of 27 dB at which $r_{so} = 80 \mu$s.

For convenience in analyzing the experimental images, we always used the same A-shaped excitation profile. Profiles with straight line elements are convenient because they are easy to inspect visually. The optimal k-space walks were searched by simulated annealing as described earlier (Serša and Macura, 1997).

VI. RESULTS

The basic properties of CARVE (influence of k-space walk, the number of events, phase uniformity, and linearity) were experimentally verified elsewhere (Serša and Macura, 1997). Here we focus on the spectral artefacts (off-resonance and relaxation) and their elimination. We also briefly consider the implementation of CARVE in three dimensions.

A. Elimination of Spectral Artefacts.

1. Manipulating the Sequence Length. Figure 7(a–c) shows a set of profiles arranged in columns recorded by the sequences shown in Figure 4(a–c). The left column shows a spectrum of the sample and the arrows indicate the position of the transmitter frequency.
Co-addition of profiles with different optimized walks improves the on-resonance/off-resonance signal ratio. (a) Four of a total of eight experimental CARVE profiles obtained with different random walks optimized to minimize gradient load. (b) Cumulative sum of all eight experimental CARVE profiles. Profiles were excited in the water/acetone sample with spectrometer frequency set to the frequency of the water line. Summation of profiles results in better profile definition due to a narrower frequency excitation window.

The profiles were obtained with a different number of experiments \( N_E \), and with a different number of pulses (events) in their respective sequences \( N_p \), but with the same set of \( N \) k-space coefficients, and with \( N_E \cdot N_p = N \). Because in all experiments the same k-space points were visited, the basic CARVE theory predicts that all the profiles are the same. However, the relaxation and the off-resonance effects make them different. The top row in Figure 7 shows the profiles of the water sample. Experiments are performed on-resonance on a relatively narrow water line (long \( T_2^\ast \)) and, thus, the profiles obtained with all three sequences are very similar. In the water/acetone sample, with the transmitter frequency still on the water line (middle row), the profile distortion is caused by the interference of the off-resonance acetone line. Off-resonance effects are superimposed on the existing on-resonance profile and show as an increase in the "noise" floor. The distortion decreases with a shortening of the sequence (from a to c) and in the single-pulse sequence is completely eliminated. Finally, when both spectral lines are off-resonance (bottom row), the profile is always blurred beyond recognition except in a single-pulse experiment. In all cases, shortening the sequence to a single event improves the quality of the experimental profile, making all single-pulse experimental profiles the same. In single-pulse experiments (column c), the out-of-profile residual signal is always the same because it originates from the limited number of k-space coefficients and not random noise.

Elimination of artefacts by shortening the sequence is achieved by reducing the time during which the phase error accumulates. Shortening the sequence increases its spectral bandwidth, making it into a single-pulse experiment broad band with a bandwidth in the range \( 2\pi f \Delta \), here 12.5 kHz.

2. Co-addition of Different Walks. The basic disadvantage of sequence shortening is that for the profile with the desired spatial resolution (defined by the total number of used k-space vectors \( N \), \( N/N_p \) experiment must be preformed. A time-saving approach, in which the spectral bandwidth can be controlled independently, is to repeat the N-pulse experiment with different random walks. Because the on-resonance profile for optimized walks is independent of the walk, and spectral artefacts do depend on the walk structure, the co-addition of profiles with different random walks results in the \( \sqrt{N_p} \) increase in the on-resonance/off-resonance signal, where \( N_p \) is the number of co-added experiments. This is illustrated in Figure 8(a) which shows four of a total of eight profiles obtained in a 100-pulse sequence with different (optimized) random walks. Figure 8(b) shows the cumulative sum of all eight profiles. Experiments were performed on the water/acetone sample with the transmitter frequency set on the water resonance (like the middle row in Figure 7). Each profile in Figure 8a represents a superposition of on-resonance and off-resonance profiles. The noise floor originates from the (beyond recognition blurred) off-resonance profile and is different in each experiment. In summing up the experimental profiles, on-resonance profiles always co-add whereas off-resonance contributions are randomly either co-add or subtract. This is clearly visible in the cumulative sums [Figure 8b)], where the quality of the resulting profile improves with the number of co-added experiments. In contrast to the previous case, elimination of spectral artefacts here is separated from the spectral bandwidth. The bandwidth is controlled by the total sequence duration (and roughly is \( 2\pi f (N \Delta t) \)) and the suppression efficacy is controlled by the number of co-added experiments, \( N_E \).

3. Compensated RF Pulses. The best approach is to compensate artefacts actively during the sequence by appropriate alteration of the RF pulse. As described by Equation (44) this is relatively straightforward for the special case where the sample spectrum is uniform. Equations (45) to (47) predict that distortions due either to relaxation, or to off-resonance, or both, can be eliminated by normalizing the original complex RF pulse with the FID intensity at appropriate time intervals. Figure 9 shows experimental verification of Equations (45) and (47) in a water/acetone sample with the transmitter frequency set between the two spectral lines. Figure 9(b) shows the experimental FID. The static magnetic field was deliberately made inhomogeneous so that the apparent transverse relaxation
time $T_2^*$ was commensurate with the sequence duration, $T_2^* = 40$ ms vs. $N\Delta \tau = 50$ ms. The actual profile obtained with an optimized (on-resonance) random walk [Figure 4(a)] is blurred beyond recognition [Figure 9(b)]. An improved profile is obtained after off-resonance correction using Equation (47) [Figure 9(c)]. An even better profile is obtained by spectral correction according to Equation (45) [Figure 9(d)]. Although the improvement is not as good as predicted, the basic advantage of the compensated pulses is that the distortions are corrected within a single experiment.

**B. Three-Dimensional CARVE.** For convenience, all aspects of CARVE are analyzed and tested in two-dimensional profiles. Besides the increased complexity of adding another dimension, there are no other restrictions in implementing the sequence in three dimensions. An extensive analysis and implementation of CARVE 3D is described elsewhere (Serca and Macura, 1998). Here we show only the basic experiment performed on-resonance for the water sample according to the sequence presented in Figure 5. The CARVE 3D sequence consists of 200 events, and the cube profile was reconstructed in a $32 \times 32 \times 32$ image. Figure 10(a) shows a standard three-dimensional image of the 5-mm water-filled tube.

Except at the top and the bottom of the tubes, all cross-sections are the same, 5-mm diameter circles. Figure 10(b) shows the calculated 200-coefficient approximation for the cube and Figure 10(c), the experimentally obtained profile.

**VII. DISCUSSION**

**A. Aim.** The purpose of the CARVE sequence is to excite an arbitrarily shaped volume with fully controlled spatial and spectral resolution. In practice, the excitation profile should be determined from the image of the studied object recorded immediately before the CARVE experiment. The sequence is custom designed for a particular shape and the desired spectrum is subsequently reordered. Because the profile properties can be calculated from selected parameters, in practice the profile does not need to be visualized. Thus, the aim of CARVE is to record one-dimensional (or two-dimensional) homonuclear (or heteronuclear) spectra with preferred properties (nuclei, spectral bandwidth, and carrier frequency) of the irregularly shaped ROI.

**B. Theory.** The principal difference between CARVE and continuous spectral spatial pulses is that for excitation, CARVE uses
discrete RF pulses (a generalized DANTE sequence) instead of a continuous RF waveform. The gradient part of CARVE can be of any shape that satisfies the following condition,

\[ γ \int_{0}^{t+1/\Delta t} G(t) \, dt = k_{i+1} - k_i \]  

(48)

The integral of the gradient over time between two successive RF pulses must be proportional to the difference between the k-space points to which these two RF pulses are assigned. This condition is very loose and allows continuous gradient sequence design similar to the one used in continuous spectral spatial pulses. As a consequence of discrete RF pulse train in CARVE k-space is visited in discrete points rather than traversed in continuous trajectories. This
introduces novel properties not common to continuous spectral spatial pulses. For example, discrete k-space points visited in CARVE can be set to a rectangular grid that matches the definition of ideal objects in k-space. This eliminates a need to regrid the continuous k-space trajectory, a common problem when continuous RF pulses are employed. The principal disadvantage of CARVE is a periodicity of its excitation profile.

Traversing k-space in hops rather than in continuous trajectories allows enormous simplification in the practical sequence implementation. For example, in the uniform excitation of the profile, each k-space point should be visited an equal number of times (normally, once). For \( N \) k-space points in discrete k-space, this can be achieved in \( N! \) equivalent ways. In the methods that traverse k-space continuously, walks with too many trajectory crossings must be avoided. Points in continuous k-space where trajectory crosses are visited twice compared to the others which result in the nonuniform weighting of the ROI. Thus, the continuous space methods are limited to the sequential or spiral walks whereas CARVE has no such limitations. Theoretically, all k-space walks are equivalent. In practice, however, they are different because the walk determines the amplitude of the gradients used. Out of \( N! \) walks, it is always possible to select and use those with desired properties.

The ability to minimize experimental distortions is very important. It implies that the theoretically predicted profile can be obtained experimentally without further corrections. A direct consequence is that in the experimental design all theoretically predicted properties of the sequence can be directly used for profile optimization. For example, the relationship between the CARVE sequence parameters and the excitation profile is linear, Equation (20). Thus, experimental profiles can be obtained in the phase-sensitive mode which enables easy implementation of the sequence distributive and additive. This is demonstrated in Figures 7 and 8 and can be further used to improve the profile properties by suitable combinations and manipulations of several profiles. For example, spatial resolution can be improved virtually without limitation by co-adding subpro- files obtained by the use of more and more k-space vectors. Because of the distributivity of the sequence, the sum of \( N_p \) profiles recorded with \( N_p \) events, using for each profile a different set of k-space vectors, generates a resulting profile with resolution equal to the \( N_p \) coefficients.

C. Controlling Spatial Resolution. In the absence of imperfections, the spatial resolution of the profile is determined exclusively by a set of selected k-space vectors (their number and distribution), Equation (14). Because the number of selected vectors must be kept reasonably low, the actual profile used is only an approximation of the targeted profile. Thus, the problem of spatial resolution is reduced to finding a suitable approximation. The squared norm used as a criterion for finding the approximation ensures that the volume (total magnetization) of the approximation is closest to the volume of the targeted profile when compared to any other combination of the same number of coefficients. However, similar volumes do not necessarily imply similar geometry. It is well known that objects with sharp edges in Fourier series representation exhibit large oscillations at the edges (Gibbs phenomenon). Thus, the sharp-edged profile approximated by the use of the second norm will have a volume very close to the targeted profile but may have oscillations that extend far from the profile borders. Thus, the spectrum recorded by CARVE may have significant contribution from magnetization components far away from the ROI. An obvious remedy for this is to change the criterion for the profile approximation, or simpler, to smooth the profile so that the transition from the ROI and the surrounding areas becomes gradual. Then, transient oscillations (if any) attenuate in the immediate vicinity of the ROI.

A more direct method of improving resolution is to use more k-space vectors. This is accomplished either by increasing the number of events in a single sequence (but only to a certain limit) or by increasing the number of experiments in which new k-space vectors are always used.

D. Controlling Spectral Bandwidth. Contrary to spatial resolution which depends only on the number (and distribution) of the k-space vectors, the spectral bandwidth of the excited profile \( \Delta \Omega \) critically depends on the duration of the CARVE sequence (roughly \( \Delta \Omega \sim 1/(N_p \Delta t) \)). Because spatial resolution depends on the total number of k-space coefficients \( N = N_p N_r \) using the distributive property of CARVE, spectral bandwidth can be controlled independently of the spatial resolution. For \( N_r = 1 \), a broad-band excitation can be achieved and for \( N_p = N \), the excitation can be narrowed to a selected spectral line. Obviously, there is a price for broad-band excitation because for a desired spatial resolution, \( N \) experiments must be performed. This, however, may not be apparent in situations where experiments need to be repeated for other reasons (e.g., when the sensitivity of the observed spectrum is low).

A narrow-band excitation can be achieved in a single experiment. However, under such conditions, the spectral bandwidth is ill defined because of the random character of the walk through k-space. The spectral bandwidth of any excitation sequence depends on its RF part. In CARVE, the RF part corresponds to a generalized DANTE sequence and its spectral profile is a convolution of the sinc function (originating from a limited duration of the sequence) and the FT of the pulse train (Morrison and Freeman, 1978). In gradient optimized CARVE sequence, the RF pulse train is random function as is the spectral profile. An obvious way to optimize the spectral profile is to add a penalty function appropriate term that defines desired properties of the RF pulse train. An alternative method to improve a definition of the spectral window is to co-add profiles of several gradient optimized walks. Then, the effective spectral profile converges toward a DANTE profile of the same duration (Morrison and Freeman, 1978). That is, the spectral window is periodic sinc function with the period \( 2\pi/\Delta \) and the approximate width \( 1/(N_p \Delta t) \).

E. Elimination of Artifacts.

1. Gradient Overload and Distortions. The most serious distortions of the excitation profile, irrespective of the method used, originate from the nonlinearity of the used gradients. Nonlinearity is usually caused by too high gradients and too fast gradient switching. In the continuous space methods, the problem is addressed either by selecting sequential or spiral trajectories or by an additional image postprocessing to correct for expected distortions. In CARVE, there are no restrictions in the way discrete k-space is traversed and thus, a walk with suitable properties can always be selected. In the example presented in Table 1, an optimized random walk has an almost six-time smaller gradient load than the unoptimized walk or a 50% smaller load than the sequential walk. One should note that even the smallest reduction of the gradient load is important since a nonlinearity is proportional to the gradient amplitude. We have chosen simulated annealing (SA) to search for the optimized walk. With our implementation of SA, it takes a few tens of seconds to find a walk with a penalty function reduced for two orders of magnitudes. In practice, the whole CARVE procedure should be interactive and the timing for calculating the CARVE parameters may be critical.
Then other protocols may be used (e.g., taboo search) or SA should be implemented more efficiently.

2. Spectral Dispersion and Relaxation. Three different methods for elimination of distortions of the CARVE profile caused by spectral distribution and relaxation have been described and demonstrated. This does not exhaust a list of possible methods but gives enough flexibility to choose or design those most suitable for particular application. For example, in complete analogy to superposition of the profiles recorded with different optimized random walks, the off-resonance effects can be eliminated by superposition of CARVE profiles obtained with different (constant) increments $\Delta f$. In addition to eliminating resonance offset, variable $\Delta f$ blurs the period of the PSF (Equation (39)) and also eliminates possible spectral foldover. Perhaps even better is to design a sequence with randomized increments $\Delta f$ within a single sequence. Of course, even better profile correction could be obtained by combining several methods. For example, one can co-add profiles acquired using different walks and compensation of the $\Theta$-coefficients according to Equation (45) to (47).

F. Carve in Three Dimensions. Practically the most important is a three-dimensional implementation of CARVE. Here, and elsewhere (Serfa and Macura, 1998), we have demonstrated that three-dimensional implementation is feasible. The main difference in three dimensions, besides adding z-axis gradient, is a large number of k-space vectors needed to obtain acceptable spatial resolution. All other aspects of the sequence design and the elimination of artefacts are the same as described for two-dimensional profiles.

VIII. CONCLUSION
We have demonstrated using simulations and actual experiments that multidimensional CARVE sequences (two and three-dimensional with desired properties can be designed. The large number of parameters that can be manipulated using the CARVE approach (number of experiments, type of penalty function for optimization of walks, number of events in the sequence, and duration of the single event) and the inherent linearity of the method enable virtually unlimited control over the spectral bandwidth and spatial resolution of the excitation profile.

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REFERENCES